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ON THE EQUATION $(x^2-1)(y^2-1)=z^2$

ABSTRACT: In this paper we get an explicit form of the formulae for all solutions in integers x,y,z of the Diophantine equation

(1) $$(x^2-1)(y^2-1)=z^2.$$

The equation (1) has been consider by K. Szymiczek [2] for the case when $x=a>1$ is a fixed integer. He proved that in this case the equation (1) has infinitely many solutions in integers x,y for every fixed integer $a>1$.

Let $T_n(u)=\cos(n \text{ arccos } u)$ be well-known Tchebyshev polynomial. In 1980 R. L. Graham [1] proved that all solutions of the equation (1) in integers x,y,z are given by the following formulae:

(2) $$x=T_n(u),\quad y=T_m(u),\quad z=\frac{1}{2}\left(T_{n+m}(u)-T_{n-m}(u)\right).$$

We note that the formulae (2) are effective but not easy to practical determination of the solutions of (1).

In this paper we prove the following theorem:

Theorem:

Let $< A_1, B_1 >$ denote the least positive solution of the Pell's equation $A^2 - DB^2 = 1$. Then all solutions of the equation (1) in the integers x, y, z are given by the formulae

$$\begin{cases} x = \frac{1}{2}\left[\left(A_1 + \sqrt{D}B_1\right)^i + \left(A_1 - \sqrt{D}B_1\right)^i\right] \\ y = \frac{1}{2}\left[\left(A_1 + \sqrt{D}B_1\right)^j + \left(A_1 - \sqrt{D}B_1\right)^j\right] \\ z = \frac{1}{4}\left[\left(A_1 + \sqrt{D}B_1\right)^i - \left(A_1 - \sqrt{D}B_1\right)^i\right]\left[\left(A_1 + \sqrt{D}B_1\right)^j - \left(A_1 - \sqrt{D}B_1\right)^j\right] \end{cases}$$

where i, j are arbitrary positive integers.

In the proof of our Theorem we use of the following Lemma.

Lemma.

Let $< A_1, B_1 >$ denote the least positive solution of the Pell's equation $A^2 - DB^2 = 1$ and let $< A_i, B_i >$denote i-th solution of this equation.

If the equation (1) has a solution in integers x, y then there exists a positive integer D such that for some i, j we have $x = A_i$ and $y = A_j$ Moreover if for every squarefree D and every i, j we take $x = A_i$ and $y = A_j$ where $< A_i, B_i >$ and $< A_j, B_j >$ are the solutions of the equation $A^2 - DB^2 = 1$ then the numbers x, y satisfy the equation (1) with uniquely determined z.

Proof.

Suppose that integers x, y, z satisfy (1). Let $(x^2 - 1, y^2 - 1) = d = Du^2$, where D denotes the squarefree kernel of d. Then we have

(3) $\quad x^2-1=dr, \quad y^2-1=ds; \quad (r,s)=1$

By (3) it follows that

(4) $\quad (x^2-1)(y^2-1)=d^2rs.$

From (1) and (4) we have $r=r_1^2$, $s=s_1^2$ and consequently

(5) $\quad x^2-1=dr_1^2=D(ur_1)^2 \quad , \quad y^2-1=ds_1^2=D(us_1)^2$

what proves first part of our Lemma.

Now, let $<A_i,B_i>$ and $<A_j,B_j>$ denote arbitrary solutions of the equation $A^2-DB^2=1$ with squarefree D. Then we have $A_i^2-1=DB_i^2$ and $A_j^2-1=DB_j^2$.

Hence $\left(A_1^2-1\right)\left(A_j^2-1\right)=\left(DB_iB_j\right)^2$. Putting $z=DB_iB_j$, $x=A_i$ and $y=A_j$ we get second part of our Lemma and the proof is complete.

Proof of the Theorem

By well-known formulae from the theory of Pell's equation and our Lemma it follows that

$$(6) \qquad \begin{cases} x=\dfrac{1}{2}\left[\left(A_1+\sqrt{D}B_1\right)^i+\left(A_1-\sqrt{D}B_1\right)^i\right] \\ y=\dfrac{1}{2}\left[\left(A_1+\sqrt{D}B_1\right)^j+\left(A_1-\sqrt{D}B_1\right)^j\right] \end{cases}$$

From (6) and (1) we obtain $z=DB_iB_j$ and

$$z=\frac{1}{4}\left[\left(A_1+\sqrt{D}B_1\right)^i-\left(A_1-\sqrt{D}B_1\right)^i\right]\left[\left(A_1+\sqrt{D}B_1\right)^j-\left(A_1-\sqrt{D}B_1\right)^j\right]$$

and the proof of our Theorem is complete.

Corollary.

Let $a > 1$ be an arbitrary fixed integer.
Then all solutions in integers y, z of the equation

$$(a^2 - 1)(y^2 - 1) = z^2$$

are given by the formulae

$$y = \frac{1}{2}\left[\left(A_1 + \sqrt{D}B_1\right)^i + \left(A_1 - \sqrt{D}B_1\right)^i\right]$$

$$z = \frac{1}{2}b\sqrt{D}\left(\left(A_1 + \sqrt{D}B_1\right)^i - \left(A_1 - \sqrt{D}B_1\right)^i\right)$$

where D is squarefree kernel of $a^2 - 1 = Db^2$ and $< A_1, B_1 >$ is the least positive integer solution of the Pell's equation $A^2 - DB^2 = 1$.

REFERENCE

[1] R. L. Graham, On a Diophantine equation arsing in graph theory, Eur. J. Comb. 1(1980), 107—122.

[2] K. Szymiczek, On some Diophantine equations connected with triangular numbers (in Polish), Zeszyty Naukowe WSP Katowice - Sekcja Mat. No 4(1964), 17—22.

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